DAMPED FREE VIBRATION EOM RESOLVING.

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With the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.6.3}$$

the traditional approach is to assume a solution of the form

$$x = e^{st} \tag{2.6.4}$$

where s is a constant. Upon substitution into the differential equation, we obtain

$$(ms^2 + cs + k)e^{st} = 0$$

which is satisfied for all values of t when

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \tag{2.6.5}$$

See 2nd LDE Article.

Equation (2.6.5), which is known as the *characteristic equation,* has two roots:

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$$s_{1,2} = -\frac{c}{2m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}_{\text{DELTA}\,\cdot\,\Delta} \tag{2.6.6}$$

Hence, the general solution is given by the equation

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$$x = Ae^{s_1 t} + Be^{s_2 t} \tag{2.6.7}$$

where A and B are constants to be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$. See the #3 page!

Equation (2.6.6) substituted into (2.6.7) gives

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$$x = e^{-(c/2m)t}\left(Ae^{\left(\sqrt{(c/2m)^2 - k/m}\right)t} + Be^{-\left(\sqrt{(c/2m)^2 - k/m}\right)t}\right) \tag{2.6.8}$$

The first term, $e^{-(c/2m)t}$, is simply an exponentially decaying function of time. The behavior of the terms in the parentheses, however, depends on whether the numerical value within the radical is positive, zero, or negative.

"Mean of causing the amplitude reduced upon time"

When the damping term $(c/2m)^2$ is larger than k/m, the exponents in the previous equation are real numbers and no oscillations are possible. We refer to this case as *overdamped.*

When the damping term $(c/2m)^2$ is less than k/m, the exponent becomes an imaginary number, $\pm i\sqrt{k/m - (c/2m)^2}\,t$. Because

~ complex number.

$$e^{\pm i\left(\sqrt{k/m - (c/2m)^2}\right)t} = \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t \pm i\sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t$$

the terms of Eq. (2.6.8) within the parentheses are oscillatory. We refer to this case as *underdamped.*

In the limiting case between the oscillatory and nonoscillatory motion, $(c/2m)^2 = k/m$, and the radical is zero. The damping corresponding to this case is called *critical damping,* c_c.

or $\omega_n = \frac{c}{2\sqrt{mK}}$ & $C_c = 2\sqrt{mK}$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km} \tag{2.6.9}$$

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Any damping can then be expressed in terms of the critical damping by a nondimensional number ζ, called the *damping ratio*:

KEY POINT OF THIS CASE!

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$$\zeta = \frac{c}{c_c} \tag{2.6.10}$$

$\zeta = 0 \rightarrow$ UNDAMPED FREE $\sim c = 0$

$\zeta < 1 \rightarrow c < 2\sqrt{mk}$

and we can also express $s_{1,2}$ in terms of ζ as follows: $\zeta = 1 \rightarrow c = 2\sqrt{mk}$; $\zeta > 1 \rightarrow c = 2\sqrt{mk}$

$$\frac{c}{2m} = \zeta\left(\frac{c_c}{2m}\right) = \zeta\omega_n$$

⚠ Key point for evaluation the system behavior.

Equation (2.6.6) then becomes

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \tag{2.6.11}$$

The three cases of damping discussed here now depend on whether ζ is greater than, less than, or equal to unity. Furthermore, the differential equation of motion can now be expressed in terms of ζ and ω_n as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}F(t) \tag{2.6.12}$$

This form of the equation for single-DOF systems will be found to be helpful in identifying the natural frequency and the damping of the system. We will frequently encounter this equation in the modal summation for multi-DOF systems.

Figure 2.6.2 shows Eq. (2.6.11) plotted in a complex plane with ζ along the horizontal axis. If $\zeta = 0$, Eq. (2.6.11) reduces to $s_{1,2}/\omega_n = \pm i$ so that the roots on the imaginary axis correspond to the undamped case. For $0 \le \zeta \le 1$, Eq. (2.6.11) can be rewritten as

$$\frac{s_{1,2}}{\omega_n} = -\zeta \pm i\sqrt{1 - \zeta^2} \quad \text{for } \zeta < 1 \tag{2.6.13}$$

The roots s_1 and s_2 are then conjugate complex points on a circular arc converging at the point $s_{1,2}/\omega_n = -1.0$. As ζ increases beyond unity, the roots separate along the

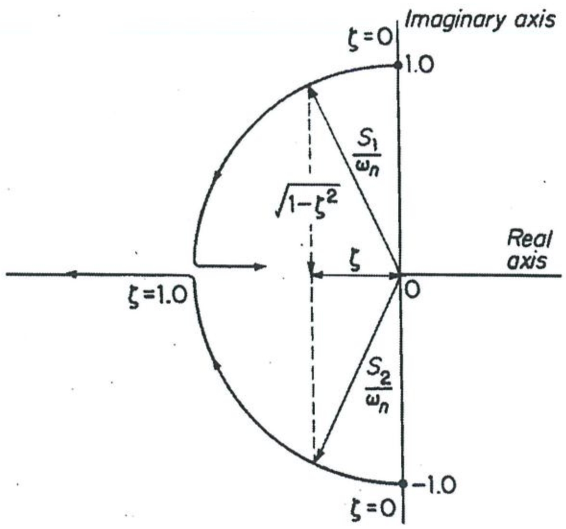

FIGURE 2.6.2.

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by used: $e^{\pm i\alpha t} = \cos\alpha t \pm i\sin\alpha t$

horizontal axis and remain real numbers. With this diagram in mind, we are now ready to examine the solution given by Eq. (2.6.8).

1 **Oscillatory motion.** [$\zeta < 1.0$ (Underdamped Case).] By substituting Eq. (2.6.11) into (2.6.7), the general solution becomes

$$x = e^{-\zeta\omega_n t}\left(Ae^{i\sqrt{1-\zeta^2}\omega_n t} + Be^{-i\sqrt{1-\zeta^2}\omega_n t}\right) \tag{2.6.14}$$

This equation can also be written in either of the following two forms:

$$x = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \sim X.e^{-\zeta\omega_n t}\sin(\omega_d + \phi) \tag{2.6.15}$$

$$= e^{-\zeta\omega_n t}\left(C_1 \sin\sqrt{1-\zeta^2}\,\omega_n t + C_2\cos\sqrt{1-\zeta^2}\,\omega_n t\right) \tag{2.6.16}$$

where the arbitrary constants X, ϕ, or C_1,C_2 are determined from initial conditions. With initial conditions $x(0)$ and $\dot{x}(0)$, Eq. (2.6.16) can be shown to reduce to

$$x = e^{-\zeta\omega_n t}\left(\underbrace{\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}}}_{C_1}\sin\sqrt{1-\zeta^2}\,\omega_n t + \underbrace{x(0)}_{C_2}\cos\sqrt{1-\zeta^2}\,\omega_n t\right) \tag{2.6.17}$$

The equation indicates that the *frequency of damped oscillation* is equal to

$$\omega_d = \frac{2\pi}{\tau_d} = \omega_n\sqrt{1-\zeta^2} \tag{2.6.18}$$

Figure 2.6.3 shows the general nature of the oscillatory motion.

2 **Nonoscillatory motion.** [$\zeta > 1.0$ (Overdamped Case).] As ζ exceeds unity, the two roots remain on the real axis of Fig. 2.6.2 and separate, one increasing and the other decreasing. The general solution then becomes

$$x = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{2.6.19}$$

where

$$A = \frac{\dot{x}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}} \tag{2.6.20}$$

PLOT

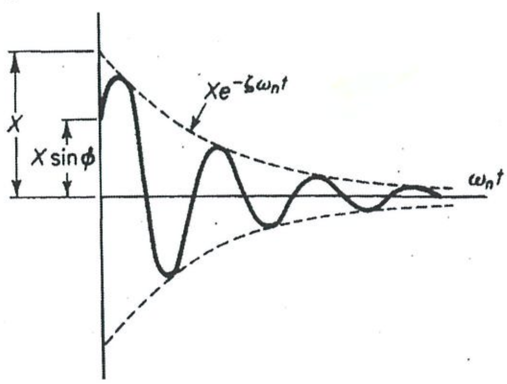

FIGURE 2.6.3. Damped oscillation $\zeta < 1.0$.

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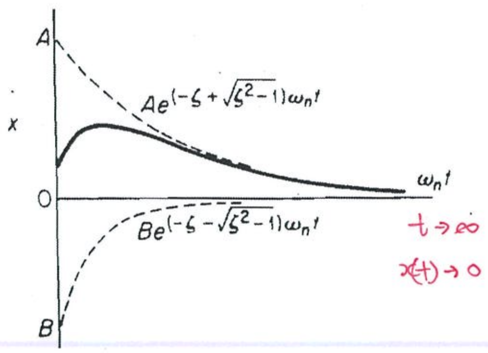

FIGURE 2.6.4. Aperiodic motion $\zeta > 1.0$.

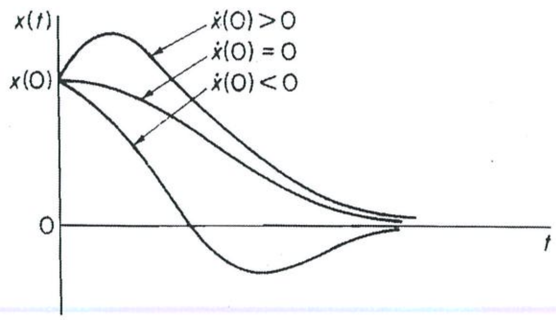

FIGURE 2.6.5. Critically damped motion $\zeta = 1.0$.

and

$$B = \frac{-\dot{x}(0) - \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{2.6.21}$$

The motion is an exponentially decreasing function of time, as shown in Fig. 2.6.4, and is referred to as *aperiodic.*

3 "**Critically" damped motion.** [$\zeta = 1.0$.] For $\zeta = 1$, we obtain a double root, $s_1 = s_2 = -\omega_n$, and the two terms of Eq. (2.6.7) combine to form a single term, which is lacking in the number of constants required to satisfy the two initial conditions.

$\Delta = $ "0" $\rightarrow$

$\Rightarrow s_{1,2} = -\frac{c}{2m} = -\zeta \cdot \omega_n$

$= -\omega_n \; (\zeta = 1)$

The correct general solution is

$$x = (A + Bt)e^{-\omega_n t} \tag{2.6.22}$$

which for the initial conditions $x(0)$ and $\dot{x}(0)$ becomes

$$x = \{x(0) + [\dot{x}(0) + \omega_n x(0)]t\}e^{-\omega_n t} \tag{2.6.23}$$

This can also be found from Eq. (2.6.17) by letting $\zeta \to 1$. Figure 2.6.5 shows three types of response with initial displacement $x(0)$.

2.7 LOGARITHMIC DECREMENT

A convenient way to determine the amount of damping present in a system is to measure the rate of decay of free oscillations. The larger the damping, the greater will be the rate of decay.

Consider a damped vibration expressed by the general equation (2.6.15)

$$x = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1 - \zeta^2}\omega_n t + \phi\right)$$

which is shown graphically in Fig. 2.7.1. We introduce here a term called the *logarithmic decrement,* which is defined as the natural logarithm of the ratio of any two successive amplitudes. The expression for the logarithmic decrement then becomes

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{e^{-\zeta\omega_n t_1}\sin\left(\sqrt{1 - \zeta^2}\omega_n t_1 + \phi\right)}{e^{-\zeta\omega_n (t_1 + \tau_d)}\sin[\sqrt{1 - \zeta^2}\omega_n(t_1 + \tau_d) + \phi]} \tag{2.7.1}$$

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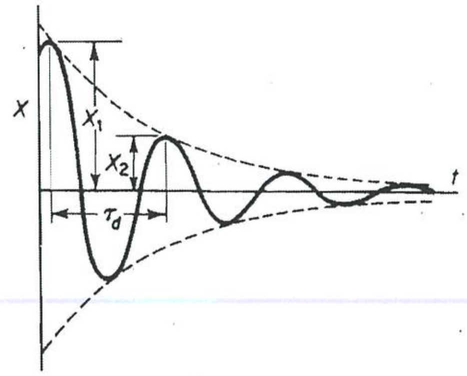

FIGURE 2.7.1. Rate of decay of oscillation measured by the logarithmic

and because the values of the sines are equal when the time is increased by the damped period τ_d, the preceding relation reduces to

$$\delta = \ln \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = \ln e^{\zeta\omega_n\tau_d} = \zeta\omega_n\tau_d \tag{2.7.2}$$

By substituting for the damped period, $\tau_d = 2\pi/\omega_n\sqrt{1-\zeta^2}$, the expression for the logarithmic decrement becomes

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.7.3}$$

which is an exact equation.

When ζ is small, $\sqrt{1-\zeta^2} \cong 1$, and an approximate equation

$$\delta \cong 2\pi\zeta \tag{2.7.4}$$

is obtained. Figure 2.7.2 shows a plot of the exact and approximate values of δ as a function of ζ.

EXAMPLE 2.7.1

The following data are given for a vibrating system with viscous damping: $w = 10$ lb, $k = 30$ lb/in., and $c = 0.12$ lb/in./s. Determine the logarithmic decrement and the ratio of any two successive amplitudes.

Solution The undamped natural frequency of the system in radians per second is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30 \times 386}{10}} = 34.0 \text{ rad/s}$$

The critical damping coefficient c_c and damping factor ζ are

$$c_c = 2m\omega_n = 2 \times \frac{10}{386} \times 34.0 = 1.76 \text{ lb/in./s}$$

$$\zeta = \frac{c}{c_c} = \frac{0.12}{1.76} = 0.0681$$

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